

1 Examples and Constructions

All of it was written by Sammy. . . I wrote nothing.

—Henri Cartan (Jackson, 1999)

Introduction. Our goal in this chapter is to construct new topological spaces from given ones. We'll do so by focusing on four basic constructions: subspaces in section 1.2, quotients in section 1.3, products in section 1.4, and coproducts in section 1.5. To maintain a categorical perspective, the discussion of each construction will fit into the following template:

- **The classic definition:** an explicit construction of the topological space
- **The first characterization:** a description of the topology as either the coarsest or the finest topology for which maps into or out of the space are continuous, leading to a better definition
- **The second characterization:** a description of the topology in terms of a universal property as given in theorems 1.1, 1.2, 1.3, and 1.4

Before we construct topological spaces, it will be good to have some examples in mind. We'll begin then in section 1.1 with examples of topological spaces and continuous maps between them.

1.1 Examples and Terminology

Let's open with examples of spaces followed by examples of continuous functions.

1.1.1 Examples of Spaces

Example 1.1 Any set X may be endowed with the *cofinite* topology, where a set U is open if and only its complement $X \setminus U$ is finite (or if $U = \emptyset$). Similarly, any set may be equipped with the *cocountable* topology whose open sets are those whose complement is countable.

Example 1.2 The empty set \emptyset and the one-point set $*$ are topological spaces in unique ways. For any space X , the unique functions $\emptyset \rightarrow X$ and $X \rightarrow *$ are continuous. The empty set is initial and the one-point set is terminal in \mathbf{Top} , just as they are in \mathbf{Set} .

Example 1.3 As we saw in section 0.1, \mathbb{R} is a topological space with the usual metric topology, but it admits other topologies, too. For example, like any set, \mathbb{R} has a cofinite

topology and a cocountable topology. The set \mathbb{R} also has a topology with basis of open sets given by intervals of the form $[a, b)$ for $a < b$. This is called the *lower limit topology* (or the *Sorgenfrey topology*, or the *uphill topology*, or the *half-open topology*). Unless specified otherwise, \mathbb{R} will be given the metric topology.

Example 1.4 In general, for any totally ordered set X , the intervals $(a, b) = \{x \in X \mid a < x < b\}$, along with the intervals (a, ∞) and $(-\infty, b)$, define a topology called the *order topology*. The set \mathbb{R} is totally ordered and the order topology on \mathbb{R} coincides with the usual topology.

Example 1.5 Unless otherwise specified, the natural numbers \mathbb{N} and the integers \mathbb{Z} are given discrete topologies, but there are others. Notably, there is a topology on \mathbb{Z} for which the sets

$$S(a, b) = \{an + b \mid n \in \mathbb{N}\}$$

for $a \in \mathbb{Z} \setminus \{0\}$ and $b \in \mathbb{Z}$, together with \emptyset , are open. Furstenberg used this topology in a delightful proof noting that there are infinitely many primes (see Mercer (2009)). It's not hard to check that the sets $S(a, b)$ are also closed in this topology, and since every integer except ± 1 has a prime factor, it follows that

$$\mathbb{Z} \setminus \{-1, +1\} = \bigcup_{p \text{ prime}} S(p, 0)$$

Since the left hand side is not closed (no nonempty finite set can be open) there must be infinitely many closed sets in the union on the right. Therefore there are infinitely many primes!

Example 1.6 Let R be a commutative ring with unit and let $\text{spec } R$ denote the set of prime ideals of R . The *Zariski topology* on $\text{spec } R$ is defined by declaring the closed sets to be the sets of the form $V E = \{p \in \text{spec } R \mid E \subseteq p\}$, where E is any subset of R .

Example 1.7 A *norm* on a real (or complex) vector space V is a function $\|\cdot\|: V \rightarrow \mathbb{R}$ (or \mathbb{C}), satisfying

- $\|v\| \geq 0$ for all vectors v with equality if and only if $v = 0$
- $\|v + w\| \leq \|v\| + \|w\|$ for all vectors v, w
- $\|\alpha v\| = |\alpha| \|v\|$ for all scalars α and vectors v

Every normed vector space is a metric space and hence a topological space with metric defined by $d(x, y) = \|x - y\|$. The standard metric on \mathbb{R}^n comes from the norm defined by $\|(x_1, \dots, x_n)\| := \sqrt{\sum_{i=1}^n |x_i|^2}$. More generally, for any $p \geq 1$, the *p-norm* on \mathbb{R}^n is defined by

$$\|(x_1, \dots, x_n)\|_p := \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}}$$

and the *sup norm* is defined by

$$\|(x_1, \dots, x_n)\|_\infty := \sup\{|x_1|, \dots, |x_n|\}$$

These norms define different metrics with different open balls, but for any of these norms on \mathbb{R}^n , the passage *norm* \rightsquigarrow *metric* \rightsquigarrow *topology* leads to the same topology. In fact, for any choice of norm on a *finite* dimensional vector space, the corresponding topological spaces are the same—not just homeomorphic but literally the same.

Example 1.8 We can generalize the previous example from \mathbb{R}^n to $\mathbb{R}^{\mathbb{N}}$, the space of sequences in \mathbb{R} , if we avoid those sequences with divergent norm. The set l_p of sequences $\{x_n\}$ for which $\sum_{n=1}^{\infty} x_n^p$ is finite is a subspace of $\mathbb{R}^{\mathbb{N}}$ (see section 1.2), and l_p with

$$\|\{x_i\}\|_p := \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{\frac{1}{p}}$$

is a normed vector space. It's difficult to compare the topological spaces l_p for different p since the underlying sets are different. For instance, $\{1/n\}$ is in l_2 but not in l_1 . Even so, the spaces l_p are homeomorphic as topological spaces (Kadets, 1967). The set l_{∞} of bounded sequences with $\|\{x_i\}\| := \sup |x_i|$ is also a normed vector space, but it is not homeomorphic to l_p for $p \neq \infty$; it is an exercise to prove it.

Note that set $\mathbb{R}^{\mathbb{N}}$ of sequences (x_1, x_2, \dots) in \mathbb{R} can be viewed as a topological space when endowed with the product topology (defined in section 1.4) and the subset of p th power summable sequences can be given a subspace topology (defined in section 1.2). The resulting topology is quite different from the topology obtained from viewing l_p as a normed vector space.

1.1.2 Examples of Continuous Functions

With some examples of spaces in tow, let's now consider a few examples of continuous functions. The first one we'll look at is a vivid illustration in **Top** of the philosophy (introduced in chapter 0) that objects are determined by their relationships with other objects.

Example 1.9 The set $S = \{0, 1\}$ with the topology $\{\emptyset, \{1\}, S\}$ is sometimes called the *Sierpiński two-point space*. In this topology, for any open set $U \subseteq X$, the characteristic function $\chi_U: X \rightarrow S$ defined by

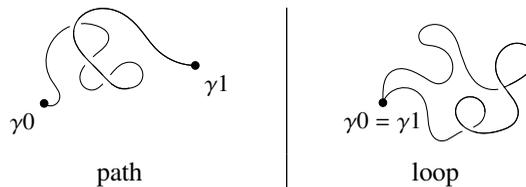
$$\chi_U(x) = \begin{cases} 1 & \text{if } x \in U \\ 0 & \text{if } x \notin U \end{cases}$$

is a continuous function. What's more, every continuous function $f: X \rightarrow S$ is of the form χ_U where $U = f^{-1}\{1\}$. Thus the open subsets of X are in one-to-one correspondence with continuous functions $X \rightarrow S$. In other words the set $\mathbf{Top}(X, S)$ is a copy of the topology of X .

Example 1.10 The previous example shows that the topology of a space X can be recovered from the set of maps $\mathbf{Top}(X, S)$, so you might wonder: *can the points be recovered, too?* But that's an easy "Yes!" Since a point $x \in X$ is the same as a map $* \rightarrow X$, the set of points of a space X is isomorphic to the set $\mathbf{Top}(*, X)$.

A practical impact of the philosophy referred to above is that a space X can be studied by looking at continuous functions either to or from a (usually simpler) space. For example, the fundamental group of X , which we'll cover in chapter 6, involves functions from the circle S^1 to X . And sequences in X , which are used to probe topological properties (as we'll see in chapter 3), are continuous functions from the discrete space \mathbb{N} to X . On the other hand, maps *out of* X are also interesting. For instance, maps from X to the discrete space $\{0, 1\}$ detect connectedness. For another instance involving maps into X , homotopy classes of maps $*$ $\rightarrow X$ reveal path components. We'll discuss both connectedness and path connectedness in section 2.1. And speaking of paths. . .

Example 1.11 A *path* in a space X is a continuous function $\gamma: [0, 1] \rightarrow X$. A *loop* in a space X is a continuous function $\gamma: [0, 1] \rightarrow X$ with $\gamma_0 = \gamma_1$.



Example 1.12 If (X, d) is a metric space and $x \in X$, then the function $f: X \rightarrow \mathbb{R}$ defined by $f(y) = d(x, y)$ is continuous.

Example 1.13 Unlike the categories Grp and $\text{Vect}_{\mathbb{k}}$ where bijective morphisms are isomorphisms, not every continuous bijection between topological spaces is a homeomorphism. For example, the identity function $\text{id}: (\mathbb{R}, \mathcal{T}_{\text{discrete}}) \rightarrow (\mathbb{R}, \mathcal{T}_{\text{usual}})$ is a continuous bijection that is not a homeomorphism. But continuous bijections are always homeomorphisms in the category of compact Hausdorff spaces—see corollary 2.18.2.

Armed with examples of topological spaces and continuous functions, we now turn to the question of construction. How can we construct new spaces from existing ones? *Subspaces, quotients, products, and coproducts* are a few ways, and although the definitions of some (or perhaps all) of these constructions may be familiar, keep in mind that the goal of the remainder of the chapter is to view them through a categorical lens. As mentioned in the chapter introduction, we'll accomplish this by exploring each construction—a subspace, a quotient, a product, a coproduct—in three stages:

- **The classic definition:** the familiar, explicit construction of the topology
- **The first characterization:** a description of the topology as either the coarsest or the finest topology for which maps in to or out of the space are continuous, leading to a better definition.
- **The second characterization:** a description of the topology in terms of a universal property

As the text unwinds, take note of the words “finest” and “coarsest” in the first characterization. Which term appears in which of the four constructions? Also keep an eye out for which topologies are characterized by maps *into* the space and which are characterized by maps *out of* the space.

1.2 The Subspace Topology

Given a set X , we can obtain a new set by choosing a subset Y of X . If X is endowed with a topology, we’d like a way to see Y as a topological space, too. This leads to the first of the four constructions—a subspace. The subspace topology is often defined (for example, in Munkres (2000)) as follows:

Definition 1.1 Let (X, \mathcal{T}_X) be a topological space and let Y be any subset of X . The *subspace topology* on Y is given by $\mathcal{T}_Y := \{U \cap Y \mid U \in \mathcal{T}_X\}$.

You can and are encouraged to check that this definition does indeed define a topology on Y . What’s more interesting, though, is the property it satisfies. In particular, Y naturally comes with an inclusion map $i: Y \rightarrow X$, and the subspace topology is the coarsest topology on Y for which i is continuous. This is its first characterization.

1.2.1 The First Characterization

Before we elaborate on this characterization, let’s consider a more general situation by way of motivation. Let (X, \mathcal{T}_X) be a topological space and let S be any set whatsoever. Consider a function

$$f: S \rightarrow X$$

It makes no sense to ask if f is continuous until S is equipped with a topology. There always exist topologies on the set S that will make f continuous—the discrete topology is one. But is there a coarser one? Is there a *coarsest* one? The answer to both questions is “yes.” Indeed, the intersection of any topologies on S for which f is continuous is again a topology on S for which f is continuous. Therefore, the intersection of *all* topologies on S for which f is continuous will be the coarsest topology for which f is continuous. Let’s call it \mathcal{T}_f and observe that it has the simple description $\{f^{-1}U \mid U \subseteq X \text{ is open}\}$. This shows that the subspace topology \mathcal{T}_Y on a subset $Y \subseteq X$ is the same as \mathcal{T}_i where $i: Y \rightarrow X$ is the natural inclusion. This prompts us to adopt a better definition for the subspace topology.

Better definition Let (X, \mathcal{T}_X) be a topological space and let Y be any subset of X . The *subspace topology* on Y is the coarsest topology on Y for which the canonical inclusion $i: Y \hookrightarrow X$ is continuous.

More generally, if S is any set and if $f: S \rightarrow X$ is an injective function, then \mathcal{T}_f —the coarsest topology on S for which f is continuous—may be called *the subspace topology* on S . This is a good definition even though the set S is not necessarily a subset of X .

Why? Since f is injective, S is isomorphic *as a set* to its image $fS \subseteq X$, and the space (S, \mathcal{T}_f) determined by $f: S \rightarrow X$ is homeomorphic to $fS \subseteq X$ with the subspace topology determined by the inclusion $i: fS \hookrightarrow X$. (If $f: S \rightarrow X$ is not injective, then there is still a coarsest topology \mathcal{T}_f on S that makes f continuous, though we won't refer to it as the *subspace* topology.)

Definition 1.2 Suppose $f: Y \rightarrow X$ is a continuous injection between topological spaces. The map f is called an *embedding* when the topology on Y is the same as the subspace topology \mathcal{T}_f induced by f .

Example 1.14 Consider the set $[0, 1]$ with the discrete topology. The evident map $i: ([0, 1], \mathcal{T}_{\text{discrete}}) \rightarrow (\mathbb{R}, \mathcal{T}_{\text{ordinary}})$ is a continuous injection, but it is not an embedding. The topology on the domain is not the subspace topology induced by i .

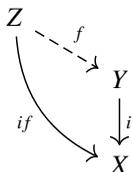
Notice that endowing a subset $Y \subseteq X$ with the subspace topology provides meaning to the question, “Is a function $Z \rightarrow Y$ continuous?” Simply put, the subspace topology determines continuous maps to Y . The converse holds as well: continuous maps to Y determine the topology on Y . This is yet another illustration of the philosophy that objects in a category are determined by morphisms to and from them. It's also the heart of the second characterization of the subspace topology.

1.2.2 The Second Characterization

This way of thinking about the subspace topology describes the important universal property which characterizes precisely which functions into the subspace are continuous—they are, reasonably, the functions $Z \rightarrow Y$ that are continuous when regarded as functions into X .

Theorem 1.1 Let (X, \mathcal{T}_X) be a topological space, let Y be a subset of X , and let $i: Y \hookrightarrow X$ be the natural inclusion. The subspace topology on Y is characterized by the following property:

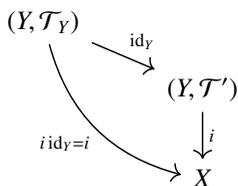
Universal property for the subspace topology For every topological space (Z, \mathcal{T}_Z) and every function $f: Z \rightarrow Y$, f is continuous if and only if the map $if: Z \rightarrow X$ is continuous.



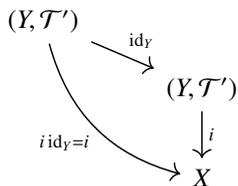
Proof. Let's think of this theorem in two parts: we'll first verify that the subspace topology has the universal property. Then we'll verify that the subspace topology is *characterized* by this universal property, which is to say that any topology on Y satisfying the universal property must be the subspace topology.

To start, let \mathcal{T}_Y be the subspace topology on Y , let (Z, \mathcal{T}_Z) be any topological space, and let $f: Z \rightarrow Y$ be a function. We have to prove that $f: Z \rightarrow Y$ is continuous if and only if $if: Z \rightarrow X$ is continuous. First, if f is continuous, then the composition of continuous functions $if: Z \rightarrow X$ is also continuous. Now suppose $if: Z \rightarrow X$ is continuous, and let U be any open set in Y . Then $U = i^{-1}V$ for some open $V \subseteq X$. Since if is continuous, the set $(if)^{-1}V \subseteq Z$ is open in Z . And since $(if)^{-1}V = f^{-1}U$, it follows that $f^{-1}U$ is open and so $f: Z \rightarrow Y$ is continuous. The topology \mathcal{T}_Y therefore has the universal property above.

Suppose now that \mathcal{T}' is any topology on Y having the universal property. We'll prove that \mathcal{T}' equals the subspace topology \mathcal{T}_Y , that is $\mathcal{T}' \subseteq \mathcal{T}_Y$ and $\mathcal{T}_Y \subseteq \mathcal{T}'$. The universal property for \mathcal{T}' is that for every topological space (Z, \mathcal{T}_Z) and for any function $f: Z \rightarrow Y$, the map f is continuous if and only if if is continuous. In particular, if we let (Z, \mathcal{T}_Z) be (Y, \mathcal{T}_Y) where \mathcal{T}_Y is the subspace topology on Y and let $f: Y \rightarrow Y$ be the identity function, then we have the following diagram



Since we know the function $i \text{id}_Y = i: Y \rightarrow X$ is continuous when Y has the subspace topology \mathcal{T} , the universal property implies that $\text{id}_Y: (Y, \mathcal{T}_Y) \rightarrow (Y, \mathcal{T}')$ is continuous, and therefore the subspace topology \mathcal{T}_Y is finer than \mathcal{T}' , that is $\mathcal{T}' \subseteq \mathcal{T}_Y$. Finally, to show that $\mathcal{T}_Y \subseteq \mathcal{T}'$, let (Z, \mathcal{T}_Z) be (Y, \mathcal{T}') and let $f = \text{id}_Y: (Y, \mathcal{T}') \rightarrow (Y, \mathcal{T}')$. So we have the following diagram



The continuity of id_Y implies that $i \text{id}_Y = i: Y \rightarrow X$ is also continuous, and so \mathcal{T}' is a topology on Y for which the inclusion $i: Y \rightarrow X$ is continuous. But the subspace topology \mathcal{T}_Y is the coarsest topology on Y for which $i: Y \rightarrow X$ is continuous, and therefore \mathcal{T}_Y is coarser than \mathcal{T}' . In other words $\mathcal{T}_Y \subseteq \mathcal{T}'$. □

Example 1.15 In the subspace topology on $\mathbb{Q} \subseteq \mathbb{R}$, open sets are of the form $\mathbb{Q} \cap (a, b)$ whenever $a < b$. Notice that the discrete and subspace topologies on \mathbb{Q} are not equivalent: for any rational r , the singleton set $\{r\}$ is open in the former but not in the latter.

1.3 The Quotient Topology

Before getting to the definition and characterizations of the quotient topology, let's recall how quotients of sets work. Suppose X is a set and let \sim be an equivalence relation on X . Then X/\sim denotes the set of equivalence classes, and the natural projection $\pi: X \rightarrow X/\sim$ that sends x to its equivalence class defines a surjective function whose fibers are the equivalence classes of \sim .

Conversely, given any set S and any surjection $\pi: X \rightarrow S$, the set S is isomorphic to X/\sim where \sim is the equivalence relation whose equivalence classes are the fibers of π :

$$x \sim y \iff \pi x = \pi y$$

The map π conveniently provides the isomorphism

$$\begin{array}{ccc} S & \xrightarrow{\cong} & X/\sim \\ s & \longmapsto & \pi^{-1}s \end{array}$$

Now suppose X is a topological space. So we have a surjective map $\pi: X \rightarrow S$ from a topological space X to a set S . What kind of topology can or should we give the set S ? This topology—called the quotient topology—is often defined as follows.

Definition 1.3 A set $U \subseteq S$ is open in the *quotient topology* if and only if $\pi^{-1}U$ is open in X .

Because S and X/\sim (the quotient determined by the fibers of π) aren't even distinguishable as sets, we can always think of the quotient topology as being defined either on S or on X/\sim . This is analogous to thinking of the subspace topology determined by an injection $f: S \hookrightarrow X$ as either being defined on S or as being defined on the subset $fS \subseteq X$.

1.3.1 The First Characterization

It doesn't make sense to ask if π is continuous when $\pi: X \rightarrow S$ is a map from a topological space X to a set S . We can ask whether there exists a topology on S for which $\pi: X \rightarrow S$ is continuous. The answer to this question is a straightforward “yes” if S is endowed with the indiscrete topology. But is there a finer topology? Is there a *finest* topology? Again, the answer is affirmative. In fact, definition 1.3 makes the quotient topology on S the finest topology for which the map $\pi: X \rightarrow S$ is continuous: declaring U to be open *only if* $\pi^{-1}U$ is open implies that π is continuous, while declaring U to be open if $\pi^{-1}U$ is open makes the quotient topology the finest topology for which π is continuous. We thus obtain the first characterization of the quotient topology, which is a better definition.

Better definition Let X be a topological space, let S be a set, and let $\pi: X \rightarrow S$ be surjective. The *quotient topology* on S is the finest topology for which π is continuous, and π is called a *quotient map*.

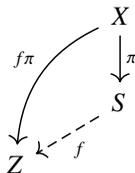
A word of caution: be careful when talking about the finest topology satisfying some property since such a topology may not exist. This is less of an issue for the *coarsest* topology satisfying a property. The difference is that the intersection of topologies is always a topology, whereas the union of topologies is usually not a topology.

1.3.2 The Second Characterization

In section 1.2.2, we observed that a topology on a subset $Y \subseteq X$ is determined by specifying what $\text{Top}(Z, Y)$ is for any space Z . Analogously, given a surjection $\pi: X \twoheadrightarrow S$ from a space to a set, a topology on S is determined by specifying what $\text{Top}(S, Z)$ is for any space Z . The universal property that characterizes the quotient topology on S tells us that the continuous maps $S \rightarrow Z$ are precisely those whose precomposition with π are continuous functions $X \rightarrow Z$.

Theorem 1.2 Let X be a topological space, let S be a set, and let $\pi: X \rightarrow S$ be surjective. The quotient topology on S is determined by the following property.

Universal property for the quotient topology For every topological space Z and every function $f: S \rightarrow Z$, f is continuous if and only if $f\pi: X \rightarrow Z$ is continuous.



Proof. Exercise. □

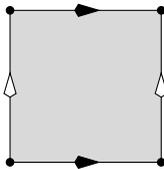
The universal property of the quotient topology tells us precisely which functions $S \rightarrow Z$ from a quotient to a space Z are continuous: they are continuous maps $X \rightarrow Z$ that are constant on the fibers of $\pi: X \rightarrow S$.

Example 1.16 The map $\pi: [0, 1] \rightarrow S^1$ defined by $\pi(t) = (\cos(2\pi t), \sin(2\pi t))$ is a quotient map. Therefore, for any space Z , continuous functions $S^1 \rightarrow Z$ are the same as continuous functions $[0, 1] \rightarrow Z$ which factor through π . That is, continuous functions $S^1 \rightarrow Z$ are the same as paths $\gamma: [0, 1] \rightarrow Z$ satisfying $\gamma 0 = \gamma 1$. These are the loops in Z .

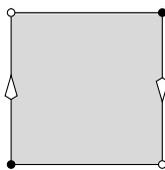
Example 1.17 The *projective space* $\mathbb{R}P^n$ is defined to be the quotient of $\mathbb{R}^{n+1} \setminus \{0\}$ by the relation $x \sim \lambda x$ for $\lambda \in \mathbb{R}$. So $\mathbb{R}P^n$ is the set of all lines through the origin in \mathbb{R}^{n+1} , and the quotient topology gives us the topology on this set of lines.

Example 1.18 As in the previous example, topological spaces are often constructed by starting with a familiar space and then identifying points to obtain a quotient. In the figure below, for instance, a new space is obtained from the unit square I^2 in \mathbb{R}^2 by identifying opposite sides. The topology on I^2/\sim is obtained from the mapping $I^2 \rightarrow I^2/\sim$ where

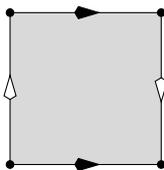
$(x, 0) \sim (x, 1)$ and $(0, y) \sim (1, y)$.



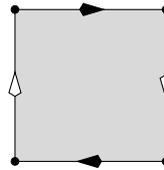
The resulting quotient space is called the *torus*, T . Other identifications of the square yield the *Möbius band* M , the *Klein bottle* K , and the *projective plane* \mathbb{RP}^2 :



M



K



\mathbb{RP}^2

Notice now that we have two definitions of \mathbb{RP}^2 : example 1.17 describes it as the space of lines through the origin, and here we've described it as a quotient of the unit square. We encourage you to verify that these two descriptions yield homeomorphic spaces.

1.4 The Product Topology

Let $\{X_\alpha\}_{\alpha \in A}$ be an arbitrary collection of topological spaces and consider the set

$$X = \prod_{\alpha \in A} X_\alpha$$

We'd like to make the set X into a topological space, but how? One way is by equipping it with the product topology, typically defined as follows.

Definition 1.4 The *product topology* on X is defined to be the topology generated by the basis

$$\left\{ \prod_{\alpha \in A} U_\alpha \mid U_\alpha \subseteq X_\alpha \text{ is open and all but finitely many } U_\alpha = X_\alpha \right\}$$

But this definition, with its surprising “all but finitely many,” suggests that there are better ways to define the product topology. And indeed, there are.

1.4.1 The First Characterization

Recall from section 0.3.3 that the set X comes with projection maps $\pi_\alpha: X \rightarrow X_\alpha$. Is there a topology on X for which these natural maps are continuous? The discrete topology is certainly one, and the intersection of all topologies that make the projections continuous will be the coarsest topology for which the projections are continuous.

Better definition Let $\{X_\alpha\}_{\alpha \in A}$ be an arbitrary collection of topological spaces and let $X = \prod_{\alpha \in A} X_\alpha$. The *product topology on X* is defined to be the coarsest topology on X for which all of the projections π_α are continuous.

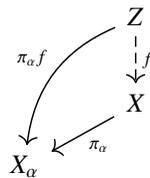
The better definition of the product topology is equivalent to definition 1.4, and we'll leave the proof as an exercise.

1.4.2 The Second Characterization

The second characterization of the product topology amounts to saying precisely which functions to the product are continuous. As before, let $\{X_\alpha\}_{\alpha \in A}$ be an arbitrary collection of topological spaces, and consider the set $X = \prod_{\alpha \in A} X_\alpha$. Keeping in mind that the universal property of the product of sets says that functions into X are the same as collections of functions into the sets X_α , it's not hard to guess that for any space Z , a map $Z \rightarrow X$ is continuous whenever all the components $Z \rightarrow X \rightarrow X_\alpha$ are continuous.

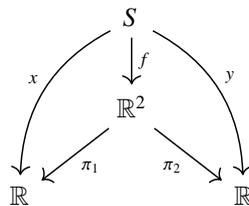
Theorem 1.3 Let $\{X_\alpha\}_{\alpha \in A}$ be an arbitrary collection of topological spaces, and let $X = \prod_{\alpha \in A} X_\alpha$. Let $\pi_\alpha : X \rightarrow X_\alpha$ denote the natural projection. The product topology on X is characterized by the following property.

Universal property for the product topology For every topological space Z and every function $f : Z \rightarrow X$, f is continuous if and only if for every $\alpha \in A$, the component $\pi_\alpha f : Z \rightarrow X_\alpha$ is continuous.



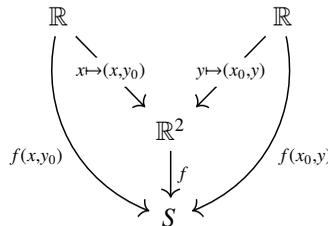
Proof. Exercise. □

Example 1.19 Let $X = \mathbb{R}^2$. One can write any function $f : S \rightarrow \mathbb{R}^2$ in terms of component functions $f s = (x s, y s)$, where the components $x s$ and $y s$ are simply given by the composition



The function f is continuous if and only if x and y are continuous, and it's good to realize that this way of specifying which functions into \mathbb{R}^2 are continuous completely determines the topology on \mathbb{R}^2 .

But be careful: functions from \mathbb{R}^2 and more generally \mathbb{R}^n can be confusing, in part because our familiarity with \mathbb{R}^n can give unjustified topological importance to the maps $\mathbb{R} \rightarrow \mathbb{R}^2$ given by fixing one of the coordinates. So don't make the mistake of thinking that a function $f: \mathbb{R}^2 \rightarrow S$ is continuous if the maps $x \mapsto f(x, y_0)$ and $y \mapsto f(x_0, y)$ are continuous for every x_0 and y_0 , as in the diagram below:



Here's a counterexample to keep in mind: the function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} \frac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

is not continuous even though for any choice of x_0 or y_0 , the maps $f(x, y_0)$ and $f(x_0, y)$ are continuous functions $\mathbb{R} \rightarrow \mathbb{R}$.

1.5 The Coproduct Topology

Let $\{X_\alpha\}_{\alpha \in A}$ be a collection of topological spaces. We'd like to make the disjoint union $X = \coprod_{\alpha \in A} X_\alpha$ into a topological space. Typically, this is done via the following explicit definition of the coproduct topology.

Definition 1.5 A set $U \subseteq X$ is open in the *coproduct topology* if and only if it is of the form $U = \coprod_{\alpha \in A} U_\alpha$ where each $U_\alpha \subseteq X_\alpha$ is open.

But since the disjoint union—when viewed as a set—comes with canonical inclusion functions $i_\alpha: X_\alpha \rightarrow \coprod X_\alpha$ for each α , we'd like a topology on X for which these natural maps are continuous.

1.5.1 The First Characterization

There are many topologies on $\coprod X_\alpha$ for which the inclusions i_α are continuous—the indiscrete topology is one. But the right topology to put on $\coprod X_\alpha$ is the finest topology for which the maps $X_\alpha \rightarrow \coprod X_\alpha$ are all continuous. This leads to another definition—one that is equivalent to definition 1.5 but better.

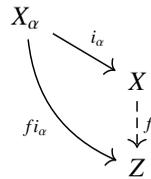
Better definition Let $\{X_\alpha\}_{\alpha \in A}$ be an arbitrary collection of topological spaces, and let $X = \coprod_{\alpha \in A} X_\alpha$. The *coproduct topology* on X is defined to be the finest topology on X for which all of the inclusions i_α are continuous.

1.5.2 The Second Characterization

To characterize the coproduct topology a second way, recall from section 0.3.3 that functions from X , viewed as a set, are determined by collections of functions from X_α . That is, for any set Z , a collection of functions $f_\alpha: X_\alpha \rightarrow Z$ corresponds uniquely to a function $X \rightarrow Z$. Not surprisingly, then, the coproduct topology on X is characterized by the following universal property.

Theorem 1.4 Let $\{X_\alpha\}_{\alpha \in A}$ be an arbitrary collection of topological spaces and let $X = \coprod_{\alpha \in A} X_\alpha$. Let $i_\alpha: X_\alpha \rightarrow X$ denote the natural inclusion. The coproduct topology on X is characterized by the following property.

Universal property for the coproduct topology For every topological space Z and every function $f: X \rightarrow Z$, f is continuous if and only if for every $\alpha \in A$, $f i_\alpha: X_\alpha \rightarrow Z$ is continuous.



Proof. Exercise. □

Example 1.20 Any set X is the coproduct over its points viewed as singletons:

$$X \cong \coprod_{x \in X} \{x\}$$

As topological spaces, however, X is homeomorphic to $\coprod_{x \in X} \{x\}$ if and only if X has the discrete topology.

Here's a summary of the results we've collected thus far:

- The *subspace topology* on a subset $Y \subseteq X$ is the *coarsest* topology for which the natural inclusion $Y \hookrightarrow X$ is continuous. It's determined by maps *into* the subspace.
- The *quotient topology* on a quotient X/\sim is the *finest* topology for which the natural projection $X \twoheadrightarrow X/\sim$ is continuous. It's determined by maps *out of* the quotient.
- The *product topology* on a set $\prod_{\alpha \in A} X_\alpha$ is the *coarsest* topology for which the natural projections $\prod_{\alpha \in A} X_\alpha \rightarrow X_\alpha$ are continuous. It's determined by maps *into* the product.

- The *coproduct topology* on a set $\coprod_{\alpha \in A} X_\alpha$ is the *finest* topology for which the natural inclusions $X_\alpha \rightarrow \coprod_{\alpha \in A} X_\alpha$ are continuous. It's determined by maps *out of* the coproduct.

You'll notice that “coarsest” and “into” are paired in the first and third constructions while “finest” and “out of” are paired in the second and fourth. This duality is no coincidence. Each of these four constructions is a special case of a more general, categorical construction—either a *limit* or a *colimit*. Limits are characterized by maps into them; colimits are characterized by maps out of them. We'll discuss these constructions in much greater detail in chapter 4, but exercises 1.12 and 1.13 at the end of the chapter provide a sneak peek.

1.6 Homotopy and the Homotopy Category

We'll close this chapter with an important application of the product topology, namely homotopy. A *homotopy* from a map $f: X \rightarrow Y$ to a map $g: X \rightarrow Y$ is a continuous function $h: X \times [0, 1] \rightarrow Y$ satisfying $h(x, 0) = fx$ and $h(x, 1) = gx$. (Notice that “the topological space $X \times [0, 1]$ ” makes sense now that we have the product topology!) Two maps $f, g: X \rightarrow Y$ are said to be *homotopic* if there is a homotopy between them, in which case we write $f \simeq g$. Homotopy defines an equivalence relation on the maps $\text{Top}(X, Y)$, the equivalence classes of which are called *homotopy classes* of maps. We use $[f]$ to denote the homotopy class of f , and we use the notation $[X, Y]$ for the set of homotopy classes of maps. You can check that the composition of homotopic maps are homotopic, and thus composition of homotopy classes of maps is well defined. The *homotopy category* of topological spaces is defined to be the category hTop whose objects are topological spaces and whose morphisms are homotopy classes of maps:

$$\text{hTop}(X, Y) := [X, Y]$$

Two spaces are said to be *homotopic* if and only if they are isomorphic in hTop . That is, X and Y are homotopic if and only if there exist maps $f: X \rightarrow Y$ and $g: Y \rightarrow X$ so that $gf \simeq \text{id}_X$ and $fg \simeq \text{id}_Y$. In this case we write $X \simeq Y$. A *homotopy invariant* is a property that is invariant under homotopy equivalence. More precisely, there is a natural functor

$$\text{Top} \rightarrow \text{hTop}$$

with $X \mapsto X$ and $f \mapsto [f]$. Functors from hTop are homotopy invariants, and functors from the category Top that factor through the functor $\text{Top} \rightarrow \text{hTop}$ are called *homotopy functors*.

Example 1.21 The space \mathbb{R}^n is homotopic to the one-point space, $\mathbb{R}^n \simeq *$. To see this, define $f: * \rightarrow \mathbb{R}^n$ by $* \mapsto 0$ and define $g: \mathbb{R}^n \rightarrow *$ in the only way possible. Then $gf = \text{id}_*$, and $fg: \mathbb{R}^n \rightarrow 0$ is homotopic to $\text{id}_{\mathbb{R}^n}$ via the homotopy $h: \mathbb{R}^n \times [0, 1] \rightarrow \mathbb{R}^n$

given by $h(x, t) = tx$. Spaces such as \mathbb{R}^n that are homotopic to a point are said to be *contractible*.

Quite often, a restricted notion of homotopy applies. For example, if $\alpha, \beta: [0, 1] \rightarrow X$ are paths from x to y , then a *homotopy of paths* is defined to be $h: [0, 1] \times [0, 1] \rightarrow Y$ satisfying $h(t, 0) = \alpha t$, $h(t, 1) = \beta t$, and $h(0, s) = x$ and $h(1, s) = y$ for all $s, t \in [0, 1]$. In other words, the homotopy fixes the endpoints of the path: for all s , the path $t \mapsto h(t, s)$ is a path from x to y agreeing with α at $s = 0$ and with β at $s = 1$. At this point, you might wonder why we're interested in fixing the endpoints of paths throughout a homotopy of paths. The reason is simple. Without this requirement, way too many paths may become homotopic: you could continuously "wind in" the endpoint of a path until it meets the initial point, then move the initial point around, then expand the point to be another path.

Succinctly put, homotopy in topology is of great importance. We'll return to it in much more detail in chapter 6.

Exercises

1. Draw a diagram of all the topologies on a three-point set indicating which are contained in which.
2. In this chapter, \mathbb{R}^n has been considered a topological space in two ways: as a metric space with the usual distance function and as the product of n copies of \mathbb{R} . Prove that these are the same.
3. Check that the Zariski topology does in fact define a topology on $\text{spec } R$, and sketch a picture of $\text{spec } \mathbb{C}[x]$ and $\text{spec } \mathbb{Z}$. For a more challenging problem, sketch a good picture of $\mathbb{Z}[x]$.
4. Give an example of a path $p: [0, 1] \rightarrow X$ connecting a to b in the space (X, \mathcal{T}) , where:

$$X = \{a, b, c, d\} \quad \text{and} \quad \mathcal{T} = \{\emptyset, \{a\}, \{c\}, \{a, c\}, \{a, b, c\}, \{a, d, c\}, X\}$$

5. Prove that any two norms on a finite dimensional vectors space (over \mathbb{R} or \mathbb{C}) give rise to homeomorphic topological spaces.
6. Prove that l_∞ is not homeomorphic to l_p for $p \neq \infty$.
7. Let $C([0, 1])$ denote the set of continuous functions on $[0, 1]$. The following define norms on $C([0, 1])$:

$$\begin{aligned} \|f\|_\infty &= \sup_{x \in [0, 1]} |f(x)| \\ \|f\|_1 &= \int_0^1 |f| \end{aligned}$$

Prove that the topologies on $C([0, 1])$ coming from these two norms are different.

8. Prove theorem 1.3. That is, prove that $X := \prod_{\alpha \in A} X_\alpha$ with the product topology has the universal property. Then prove that if X is equipped with any topology having the universal property, then that topology must be the product topology.
9. Are the subspace and product topologies consistent with each other?

Let $\{X_\alpha\}_{\alpha \in A}$ be a collection of topological spaces, and let $\{Y_\alpha\}$ be a collection of subsets; each $Y_\alpha \subseteq X_\alpha$. There are two things you can do to put a topology on $Y := \prod_{\alpha \in A} Y_\alpha$:

- a) You can take the subspace topology on each Y_α , then form the product topology on Y .
- b) You can take the product topology on X , view Y as a subset of X , and equip it with the subspace topology.

Is the outcome the same either way? If yes, prove it using only the universal properties. If no, give a counterexample.

10. Prove that the quotient topology is characterized by the universal property given in section 1.3.
11. Are the quotient and product topologies compatible with each other?

Let $\{X_\alpha\}_{\alpha \in A}$ be a collection of topological spaces, let $\{Y_\alpha\}_{\alpha \in A}$ be a collection of sets, and let $\{\pi_\alpha: X_\alpha \rightarrow Y_\alpha\}_{\alpha \in A}$ be a collection of surjections. Let $X = \prod_\alpha X_\alpha$; notice that you have a surjection $\pi: X \rightarrow Y$. There are two ways to put a topology on $Y := \prod_{\alpha \in A} Y_\alpha$:

- a) Take the quotient topology on each Y_α , then form the product topology on Y .

b) Take the product topology on X , then put the quotient topology on Y .

Is the outcome the same either way? If yes, prove it using only the universal properties. If no, give a counterexample.

12. Suppose X is a topological space and $f: X \rightarrow S$ is surjective. Define an equivalence relation on X by $x \sim x' \iff fx = fx'$. Let

$$R = \{(x, x') \in X \times X \mid fx = fx'\}$$

There are two maps, call them $r_1: R \rightarrow X$ and $r_2: R \rightarrow X$, defined by the composition of inclusion $R \hookrightarrow X \times X$ with the two natural projections $X \times X \rightarrow X$.

$$R \hookrightarrow X \times X \begin{array}{c} \xrightarrow{\pi_1} \\ \xrightarrow{\pi_2} \end{array} X$$

Learn what a coequalizer is and prove that the set S with the quotient topology is the coequalizer of r_1 and r_2 .

13. Suppose X is a topological space and $f: S \hookrightarrow X$ is injective. Let X/\sim be the quotient space generated by the equivalence relation $x \sim y \iff x, y \in fS$. Then there is a diagram:

$$X \begin{array}{c} \xrightarrow{\pi} \\ \xrightarrow{c} \end{array} X/\sim$$

where π is the natural projection sending an element to its equivalence class and c is the constant map sending each $x \in X$ to the equivalence class of fS . Learn what an equalizer is and prove that the set S with the subspace topology is the equalizer of π and c .

14. This exercise starts off with a definition:

Definition 1.6 Let X and Y be topological spaces. A function $f: X \rightarrow Y$ is called *open* (or *closed*) if and only if fU is open (or closed) in Y whenever U is open (or closed) in X .

Let (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) be topological spaces, and suppose $f: X \rightarrow Y$ is a continuous surjection.

- a) Give an example to show that f may be open but not closed.
- b) Give an example to show that f may be closed but not open.
- c) Prove that if f is either open or closed, then that the topology \mathcal{T}_Y on Y is equal to \mathcal{T}_f , the quotient topology on Y .

15. Consider the closed disc D^2 and the two-sphere S^2 :

$$\begin{aligned} D^2 &= \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\} \\ S^2 &= \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\} \end{aligned}$$

Consider the equivalence relation on D^2 defined by identifying every point on $S^1 \subseteq D^2$. So each point in $D^2 \setminus S^1$ is a one point equivalence class, and the entire boundary ∂D^2 is one equivalence class. Prove that the quotient D^2/\sim with the quotient topology is homeomorphic to S^2 .